LMI BASED CONTROL OF SATELLITE SYSTEM

Andrey Yonchev [1)]
[1)] Technical University of Sofia, Faculty of Automation, ELDE, 8 Kliment Ohridski, 1000 Sofia, Bulgaria, E-mail: ajonchev@mail.bg

Abstract: Many problems existing in control theory can be effectively solved using Linear Matrix Inequalities (LMIs). Solutions of these inequalities are obtained utilizing semi-definite programming as a generalization of the linear programming. In this paper we consider the problem of stabilizing multivariable state feedback controllers, which stabilize the satellite system. First quadratic stability problem via introducing Lyapunov functions is considered. Then using LMIs bounded energy problem is considered and finally quadratic H_∞ control ensuring closed loop performance γ is realised. Stability and performance specifications are transformed in terms of LMIs. Numerical examples are also presented.

Key words: Lyapunov function, LMI-based synthesis, bounded energy problem, H_∞ control

1. Introduction

The satellite system is a popular laboratory tool serving as a benchmark for testing linear and nonlinear control techniques. It is a electromechanical device having complex dynamic behaviour. The satellite system is adapted from [1] system can be used as an impressive demonstration tool for missile stabilization problems. The system is a satellite consisting of two rigid bodies (main body and instrumentation module) joined by a flexible link (the "boom"). The boom is modeled as a spring with torque constant and viscous damping.

The goal of this paper is to derive stabilizing multivariable state feedback controllers using the LMI technique. First quadratic stability via introducing Lyapunov functions has to be analyzed. Then bounded energy problem has to be considered and finally quadratic H_∞ control ensuring closed loop performance γ has to be realized. Stability and performance specifications have to be transformed in terms of LMIs.

The effectiveness of LMI approach remains valuable for several reasons. To begin with it is applicable to all plants without restrictions on infinite or pure imaginary invariant zeros. In addition LMI based design is practical and interesting thanks to the availability of efficient convex optimization algorithms [2] and software [3] plus the MATLAB package Yalmip and SeDuMi solver [4].

The remainder of the paper is organized as follows. In Section 2 we shortly present the problem set up and objective. Section 3 describes the LMI based approach to design multivariable state-feedback controllers. Section 4 presents some numerical examples before we conclude in Section 5 with some final remarks.

2. Problem Set up and Objective

2.1 Satellite System

Consider the free body diagram of the satellite system, shown on Figure 1. The considered dynamical system is of forth order with state variables as follows: yaw angle for the main body - θ_1 and yaw angle for the sensor module - θ_2, angular velocity for the main body - $\dot{\theta}_1$, and the corresponding angular velocity for the sensor module - $\dot{\theta}_2$. The input signal is T - the control torque, k - torque constant, f - viscous damping. The numerical values of the parameters are given in Table1.

k	f
0.245	0.0219

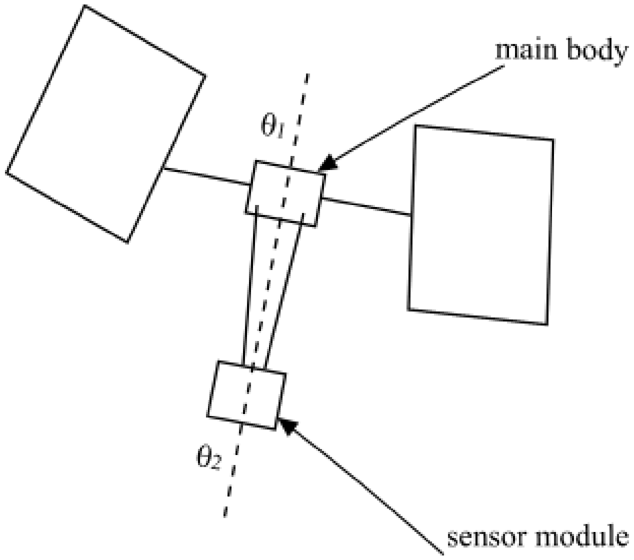

Fig.1: Satellite system

The state space model of the system by choosing the state variables $x_1 = \theta_1$, $x_2 = \theta_2$, $x_3 = \dot{\theta}_1$, $x_4 = \dot{\theta}_2$ is

$$A=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k & k & -f & f \\ k & -k & f & -f \end{bmatrix}, B=\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, C=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad D=\begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

2.2 LMI based control

The considered satellite system is unstable. That is why in this paper we design stabilizing multivariable state feedback controllers using the LMI technique. In the beginning quadratic stability via introducing Lyapunov function has to be analyzed. Then bounded energy problem has to be solved in order to ensure optimal control of the considered system and finally quadratic H_∞ control ensuring closed loop performance γ has to be realized. Stability and performance specifications have to be transformed in terms of LMIs. The objective of the paper is also connected with showing efficiency and effectiveness of the LMI based control. More information about LMI based techniques can be obtained in [5, 6, 7].

3. LMI based state – feedback control

3.1 LMI based quadratic stability problem

In this subsection we use the LMI technique to design a multivariable state-feedback controller in order to ensure quadratic stability. This means that every state trajectory should converge to 0 as time goes to infinity.

For linear time-invariant systems (LTI) given in state-space form – open loop system

$$\dot{x} = Ax + Bu \tag{1}$$

it is necessary to find a state-feedback matrix K such that system (1) should be stable. The control input is chosen as $u=Kx$ then the closed loop system is obtained

$$\dot{x} = (A + BK)x \ . \tag{2}$$

To make the LTI system stable it is necessary to use a quadratic Lyapunov function

$$V(x) = x^T Px, \quad P > 0, \quad P = P^T \tag{3}$$

such that

$$\frac{d}{dt}V(x) = \dot{x}^T Px + x^T P\dot{x} = x^T[(A+BK)^T P + P(A+BK)]x < 0 \tag{4}$$

In order to ensure quadratic stability of the system (1) the following system of inequalities has to be solved

$$(A+BK)^T P + P(A+BK) < 0, \quad P > 0. \tag{5}$$

The system of inequalities is nonlinear with respect to the unknowns P and K that is why we perform linearizing change of variables

$$Q = P^{-1} \Rightarrow P = Q^{-1}, \quad Y = KP^{-1} = KQ \Rightarrow K = YQ^{-1},$$

to obtain

$$A^T P + K^T B^T P + PA + PBK < 0,$$
$$A^T Q^{-1} + Q^{-1} Y^T B^T Q^{-1} + Q^{-1} A + Q^{-1} BYQ^{-1} < 0.$$

Finally we multiply on left and right the last inequality with Q to obtain a system of LMIs with respect to Q and Y

$$AQ + QA^T + Y^T B^T + BY < 0, \quad Q > 0. \tag{6}$$

3.2 LMI based bounded energy problem

For the open loop LTI system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du, \end{aligned} \tag{7}$$

the bounded problem means to find a control law which minimizes the output energy for a given initial condition *x(0)*

$$\int_0^\infty y^T y dt.$$

The control input is chosen as $u=Kx$ then the closed loop system is obtained

$$\dot{x} = (A+BK)x,$$
$$y = (C+DK)x.$$

To make the LTI system stable using bounded energy problem it is necessary to use a quadratic Lyapunov function $V(x) = x^T Px, \quad P > 0$ such that

$$\frac{d}{dt}V(x) = x^T[(A+BK)^T P + P(A+BK)]x \le -y^T y. \quad (8)$$

Finally to realize the LMI based bounded energy problem one should solve the following system of inequalities

$$x^T[(A+BK)^T P + P(A+BK)]x \le -y^T y, \quad P > 0. \quad (9)$$

This system of inequalities is nonlinear with respect to the unknowns P and K that is why we perform linearizing change of variables to obtain

$$AQ + QA^T + BY + Y^T B^T + (CQ+DY)^T(CQ+DY) \le 0, \quad Q > 0. \quad (10)$$

Since the upper LMI is still nonlinear with respect to Q and Y we have to use the Schur complement argument [8] to obtain the following system of LMIs

$$Q > 0$$
$$\begin{bmatrix} AQ + QA^T + BY + Y^T B^T & (CQ+DY)^T \\ (CQ+DY) & -I \end{bmatrix} \le 0. \quad (11)$$

3.3 LMI based quadratic H_∞ performance problem

From [6] we know that the LTI system (7) is asymptotically stable if the following statements are true:

- $\|G(s)\|_\infty < \gamma, \quad$ (12a)
- for every input signal u and initial condition *x(0)=0*

$$\sup_{0<\|u\|_2<\infty} \frac{\|y\|}{\|u\|} < \gamma, \quad (12b)$$

- there exists a solution of the following LMIs

$$\begin{bmatrix} A^T P + AP + C^T C & PB + C^T D \\ B^T P + D^T C & D^T D - \gamma^2 I \end{bmatrix} \le 0, \quad P > 0. \quad (12c)$$

It is necessary to obtain a quadratic Lyapunov function $V(x) = x^T Px$ and a scalar $\gamma > 0$ such that for every t the following inequality holds

$$\frac{d}{dt}V(x) + y^T y - \gamma^2 u^T u \le 0. \quad (13)$$

If we integrate from *0* till *T* for *x(0)=0* we obtain

$$V(x(T)) + \int_0^T (y^T y - \gamma^2 u^T u) dt \le 0. \quad (14)$$

Since $V(x(T)) \ge 0$ thus we have

$$\int_0^T y^T y dt - \gamma^2 \int_0^T u^T u dt \le 0,$$
$$\|y\|_2^2 - \gamma^2 \|u\|_2^2 \le 0,$$
$$\frac{\|y\|_2}{\|u\|_2} \le \gamma.$$

The expression (13) can be represented as (12c) according to [5], which is actually the Eigenvalue Problem with respect to the variables P and γ.

To obtain quadratic H_∞ performance it is also necessary to ensure that (12a) holds. The transfer function matrix of system (7) is given by the expression

$$G(s) = C(sI - A)^{-1} + D.$$

And the H_∞ norm is defined as $\|G(s)\|_\infty = \sup_{\omega \in R} \sigma_{\max}(G(j\omega))$. Now for the unstable open loop system (7) with $D=0$ we would like to design a controller K such that for the corresponding closed loop system quadratic stability and performance γ has to be ensured using the LMI approach. Using the Schur complement argument expressions (13) and

(12c) are represented via the following inequality with unknowns K, P and γ

$$\gamma > 0,$$
$$P > 0,$$
$$\begin{bmatrix} (A+BK)^T P + P(A+BK) & 0 & C^T \\ 0 & -\gamma I & 0 \\ C & 0 & -\gamma I \end{bmatrix} \leq 0.$$

Unfortunately the upper inequality is nonlinear with respect to the variables K and P so a linearizing change has to be performed. This will help us obtain the LMI system suitable for calculating a controller K such that for the considered closed loop system quadratic stability and performance γ can be ensured

$$\gamma > 0, \tag{15}$$
$$Q > 0,$$
$$\begin{bmatrix} (AQ+BY)^T + (AQ+BY) & 0 & QC^T \\ 0 & -\gamma I & 0 \\ CQ & 0 & -\gamma I \end{bmatrix} \leq 0.$$

4. Numerical Examples

Consider the satellite system given in state - space model – *A, B, C, D* – Section 2. The open loop system is unstable since the eigen values of matrix A are -0.0000, -0.0219 + 0.6997i, -0.0219 - 0.6997i, 0. In Fig. 2 is shown the time response of the open loop system

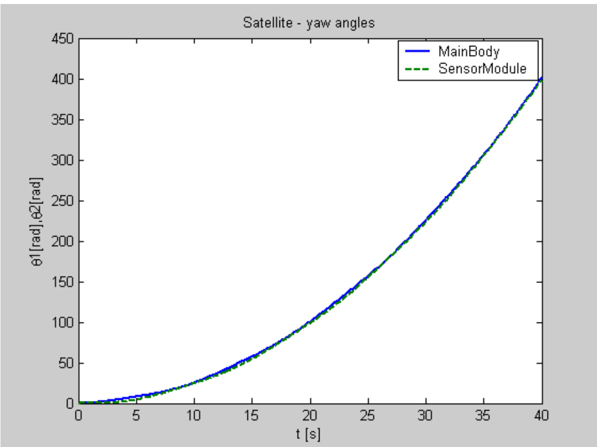

Fig.2: Time response of the open loop system

4.1 LMI based quadratic stability problem

Using inequality (6) and applying MATLAB package Yalmip and SeDuMi solver the controller matrix

$K = YQ^{-1} = [-1.6616 \quad -0.1922 \quad -1.1288 \quad -1.6594]$

can be obtained. The closed loop system is then stable since the eigenvalues of A-BK are

-0.4425 + 1.2221i, -0.4425 - 1.2221i
-0.1438 + 0.4982i, -0.1438 - 0.4982i.

Fig. 3 shows the time response of the closed loop system with K.

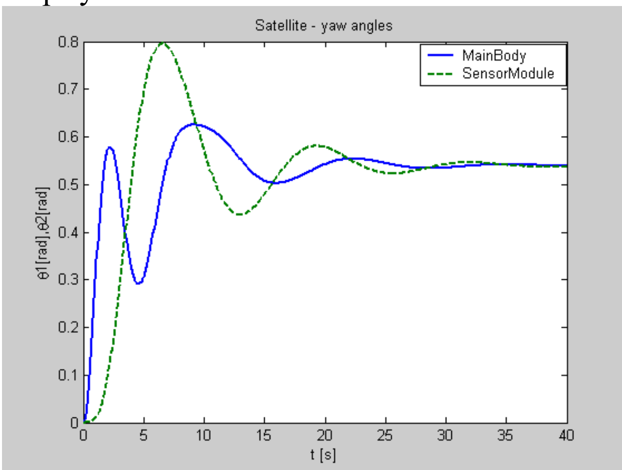

Fig.3: Time response of the closed loop system with K

4.2 LMI based bounded energy problem

Using inequality (11) and applying MATLAB package Yalmip and SeDuMi solver the controller matrix

$K\_bep = Y\_bepQ^{-1}\_bep$=[-4.6753 -1.1800 - 1.8739 -7.7270]

can be obtained. The closed loop system is then stable since the eigenvalues of A-BK_bep are

-0.7303 + 1.9614i, -0.7303 - 1.9614i
-0.2285 + 0.5247i, -0.2285 - 0.5247i.

Fig. 4 shows the time response of the closed loop system with K_bep.

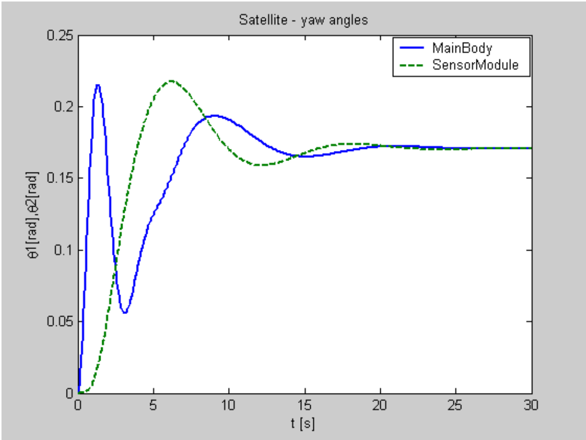

Fig.4: Time response of the closed loop system with *K_bep*.

4.3 LMI based quadratic H_∞ performance problem

Using inequality (15) and applying MATLAB package Yalmip and SeDuMi solver the controller matrix

$K\_h = Y\_hQ^{-1}\_h$=[-4.2910 -0.9983 -1.8030 -7.0643]

can be obtained. The closed loop system is then stable since the eigenvalues of *A-BK_h* are

-0.6943 + 1.8799i, -0.6943 - 1.8799i
-0.2291 + 0.5198i, -0.2291 - 0.5198i.

Fig. 5 shows the time response of the closed loop system with *K_h*.

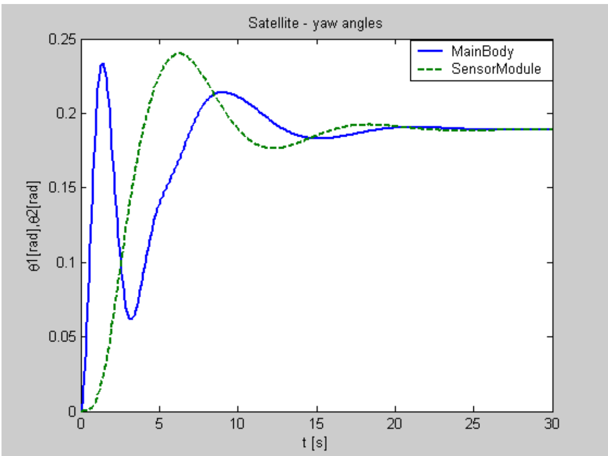

Fig.5: Time response of the closed loop system with *K_h*

5. Conclusions

This paper considers the problem of designing stabilizing multivariable state feedback controllers for the satellite system using the LMI technique. Applying Lyapunov functions quadratic stability is assured. Then bounded energy problem is solved and finally quadratic H_∞ control ensuring closed loop performance γ is realized. Stability and performance specifications are transformed in terms of LMIs. Based on these results we have presented numerical examples to explicitly reveal the performance and applicability of the LMI approach to design multivariable state-feedback controllers for LTI systems.

Acknowledgement
The research results presented in this paper are funded by Technical University – Sofia research project No. 102ни046-8.

References:

[1] J. C. Doyle, K. Khargonekar, and B. Francis. *State-space solutions to Standard H_2 and H_∞ Control Problems,* IEEE Trans. Aut. Contr., AC-34, pp. 831-847, 1989.
[2] Y. Nestorov and P. Gahinet, *Interior-Point Polynomial Algorithms in Convex Programming*, SIAM, Philadelphia, PA,1994.
[3] P. Gahinet, A. Nemirovski, A. Laub, and M. Chilali. *LMI Control Toolbox for Use with MATLAB.* The MathWorks, Inc., 2000
[4] D. Paucelle, D. Henrion, Y. Labit, and K. Taitz. *User's Guide for SeDuMi Intergace 1.04.* LAAS-CNRS, 2002.
[5] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory.* SIAM, Philadelphia, 1994.
[6] C. Sherer, and S. Weiland. *Linear Matrix Inequalities in Control.* Eindhoven University of Technology, Eindhoven, 2004.
[7] A. Yonchev, M. Konstantinov, and P. Petkov. *Linear Matrix Inequalities in Control Theory,* Demetra, Sofia, 2005. ISBN 954:9526-32-1 (in Bulgarian).
[8] G. E. Dullerud, and F. Paganini. *A Course in Robust Control Theory.* Springer-Verlag, New York, 2000.